Remarks on "Note about Hamiltonian formalism of healthy extended Hořava-Lifshitz gravity" by J. Klusoň

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Abstract

We reassess the conclusion by Klusoň (*J. High Energy Phys. 1007 (2010) 038*) that the Hamiltonian formulation of the healthy extended Hořava-Lifshitz gravity does not present any problem.

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I. INTRODUCTION

In note [1] and its companion note [2] the author applies the Hamiltonian formalism to the actions of the so-called healthy extension of the Hořava and Hořava-type models [3]. The healthy extension was proposed in [4] where, in particular, it was claimed that the canonical structure (Hamiltonian formulation) of Hořava-type actions supplemented by the healthy extension does not present any problem, contrary to the occurrence of pathologies in the Hamiltonian formulation of Hořava's original proposal for the projectable and non-projectable cases (e.g. see [5–8]). Both papers [1, 2] confirm the assertion of [4] about the health of the healthy extension in the Hamiltonian formalism; we shall reassess this conclusion.

II. RE-EXAMINATION OF KLUSOŇ'S HAMILTONIAN ANALYSIS

A. Brief review of Klusoň's notation and results

The Hamiltonian formulation of the healthy extended Hořava models was considered in [1] and for a slightly more complicated model in [2]; the latter produces nothing peculiar in comparison with the simpler model. The action given by

$$S\left(N, N^i, g_{km}\right) = \int dtd^Dx\sqrt{g}N\left(K_{ij}G^{ijkl}K_{kl} - E^{ij}G_{ijkl}E^{kl} - V\left(g_{ij}, a_i\right)\right) \tag{1}$$

(see Eq. (2.3) of [1]), where

$$K_{ij} = \frac{1}{2N}\left(\dot{g}_{ij} - \nabla_i N_j - \nabla_j N_i\right), \tag{2}$$

$$E^{ij} = \frac{\delta W}{\delta g_{ij}}, \tag{3}$$

and G_{ijkl} is inverse of

$$G^{ijkl} = \frac{1}{2}\left(g^{ik}g^{jl} + g^{il}g^{jk}\right) - \lambda g^{ij}g^{kl} \tag{4}$$

for $\lambda \neq \frac{1}{D}$, where D is spatial dimension (more details can be found in any article on Hořava-type models). The covariant derivative with respect to the spatial metric is denoted by ∇_j. The potential $V\left(g_{ij}, a_i\right)$ was not specified in either [1] or [2], but the author refers to the paper in which it was introduced (see Eq. (6) of [4]), where the particular choice of terms

was also explained. We reproduce the equation for $V\left(g_{ij},a_i\right)$ here:

$$V\left(g_{ij},a_i\right)=-\alpha g^{ij}a_ia_j+M_P^{-2}\left(C_1a_i\triangle a^i+C_2\left(a_ia^i\right)^2+C_3a_ia_jR^{ij}+...\right)+ \qquad (5)$$

$$M_P^{-4}\left(D_1a_i\triangle^2a^i+D_2\left(a_ia^i\right)^3+D_3a_ia^ia_ja_kR^{jk}+...\right)$$

with

$$\triangle=g^{ij}\nabla_i\nabla_j \qquad (6)$$

and

$$a_i\equiv\frac{\partial_iN}{N}=\partial_i\ln N. \qquad (7)$$

Note: although a_i is sometimes called an "additional vector"[2], it is only a short notation for combination (7), not an independent variable. This combination is a "D-dimensional vector"[1] or "3-vector" in the original paper [4].

The first steps of the Hamiltonian formulation are standard and were followed in [1]. By performing the Legendre transformation,

$$H=\dot{N}p_N+\dot{N}^ip_i+\dot{g}_{ij}p^{ij}-L,$$

where p_N, p_i, p^{ij} are the momenta conjugate to all independent variables N, N^i, g_{ij} of (1), and expressing the velocities in terms of momenta, one obtains the following total Hamiltonian[1]:

$$H_T=\int d^Dx\left[N\left(\mathcal{H}_\perp+\sqrt{g}V\right)+N^i\mathcal{H}_i+\dot{N}^ip_i+\dot{N}p_N\right], \qquad (8)$$

where p_i and p_N are primary constraints, and

$$\mathcal{H}_\perp=\frac{1}{\sqrt{g}}p^{ij}G_{ijkl}p^{kl}+\sqrt{g}E^{ij}G_{ijkl}E^{kl}, \qquad (9)$$

$$\mathcal{H}_i=-2g_{ik}\partial_jp^{kj}-\left(2\partial_jg_{ik}-\partial_ig_{jk}\right)p^{jk}. \qquad (10)$$

[1] A non-standard definition is used in [1] – "the total Hamiltonian is the sum of the original Hamiltonian and all constraints"; in the literature on constrained dynamics such a combination is called an extended Hamiltonian, but this extension is not healthy because equivalence with the Lagrangian is lost (see [9]). For GR, the total Hamiltonian is a linear combination of constraints, which is a common feature of generally covariant theories.

Note that $\mathcal{H}_{\perp}$ is the same as the so-called Hamiltonian constraint[2] of the unhealthy Hořava model, and $\mathcal{H}_i$ coincides with the momentum constraint for GR in ADM variables. The time development of the primary constraints leads to

$$\dot{p}_i = \{p_i, H_T\} = -\mathcal{H}_i \approx 0, \tag{11}$$

$$\dot{p}_N = \{p_N, H_T\} = -\mathcal{H}_{\perp} - \sqrt{g}V + \frac{1}{N}\partial_i \left(N\sqrt{g}\frac{\delta V}{\delta a_i} \right) \equiv \Theta_2 \approx 0. \tag{12}$$

Further, the consideration of the time development of (12),

$$\dot{\Theta}_2 = \{\Theta_2, H_T\} = \left\{\Theta_2, N\left(\mathcal{H}_{\perp} + \sqrt{g}V\right) + N^i \mathcal{H}_i\right\} + \left\{ -\sqrt{g}V + \frac{1}{N}\partial_i \left(N\sqrt{g}\frac{\delta V}{\delta a_i} \right), \dot{N}p_N \right\},$$

gives a non-linear partial differential equation for undetermined velocity, $\dot{N}$ (the Lagrange multiplier). The primary, p_N, and secondary, Θ_2, constraints were classified in [1] as a second-class pair, and this fact was presented as a key advantage of the healthy extension because it allows one to perform the Hamiltonian reduction, i.e. go to the reduced phase-space by eliminating the pair of canonical variables (N, p_N): $p_N = 0$ and $N = N\left(\mathcal{H}_{\perp}, g_{km}\right)$ (the solution to $\Theta_2 = 0$ for N as a function of remaining phase-space variables).

We note that solving $\Theta_2 = 0$ of (12) is not the same as finding a solution to the second-class constraints in known field-theoretical examples, e.g. first-order formulation of the Maxwell, Yang-Mills theory, and affine-metric formulation of GR, where such equations are algebraic with respect to the eliminated fields, and the reduction can be performed either at the Lagrangian or Hamiltonian levels, with the same outcome (e.g. see [10]). This important difference was not commented on or even noticed in [1, 2], and an assumption was made that the solution to (12) can be found. The appearance of differential second-class constraints[3] is a common feature of Hamiltonian formulations of Hořava-type models, but to the best of our knowledge, this fact was mentioned only in a paper by Pons and Talavera [7] (see footnote 3), without elaboration on the effect or possible consequences of solving such second-class constraints in the Hamiltonian reduction; it is merely mentioned, that unlike mechanics, where the multipliers are "undoubtedly determined", this is "not the

[2] In [1] a different notation is used for our $\mathcal{H}_{\perp}$ (i.e. $\mathcal{H}_T$), which is somewhat confusing since the standard use of subscript, $_T$, is for the total Hamiltonian.

[3] In field theories, differential constraints are not something unusual, but here "differential" manifests not just a presence of some derivatives, but derivatives of fields with respect to which these constraints should be solved.

case in field theory". Therefore, for differential constraints like (12), a multiplier cannot be uniquely determined; the term "the partial determination of the multiplier" is used in [7]. If a multiplier is partially determined, the constraint is only partially second-class; therefore, it is also partially first-class, and the Hamiltonian reduction can (if at all) only be partially performed. Problems of this sort, as well as the problems in calculating the Dirac Brackets for such partial second-class constraints, are discussed in [11] (see Section 4 of Chapter 2); but an undoubted solution to these problems is unknown, and should be found before any conclusion can be drawn about formulations that have differential (partially) second-class constraints. If neglected, these problems will reappear at the later stages of the Hamiltonian procedure. To demonstrate this fact we shall continue our analysis, and like the author of [1, 2], assume that such constraints have no peculiarities, and we return to the equation:

$$-\mathcal{H}_{\perp}-\sqrt{g}V+\frac{1}{N}\partial_i\left(N\sqrt{g}\frac{\delta V}{\delta a_i}\right)=0. \tag{13}$$

Note: equation (13) is a complicated, nonlinear partial differential equation of higher order (e.g. considering the term with $\triangle^2$ in (5) makes this equation of sixth order in spatial derivatives of lapse); and the existence and uniqueness of a solution deserves careful analysis. As a second-class constraint, (13) is assumed to have a solution that allows one to perform the Hamiltonian reduction [1], i.e. one can find $N = N\left(\mathcal{H}_{\perp}, g_{km}\right)$ (express one canonical variable in terms of others). After such an elimination it is also necessary to find the Dirac Brackets (again, not a unique operation for differential constraints (see [11])); but if one neglects these subtle questions and just mimics the case of mechanical models, or algebraic constraints in field theory, then the result follows immediately – the Dirac Brackets, for the variables remaining after reduction, are the same as the Poisson Brackets (PBs).

The solution of the second-class constraints (8), i.e. $p_N = 0$ and $N = N\left(\mathcal{H}_{\perp}, g_{km}\right)$, must be substituted into Hamiltonian; and the reduced total Hamiltonian follows,

$$H_T=\int d^Dx\left[N\left(\mathcal{H}_{\perp},g\right)\left(\mathcal{H}_{\perp}+\sqrt{g}V\left(\mathcal{H}_{\perp},g\right)\right)+N^i\mathcal{H}_i+\dot{N}^ip_i\right] \tag{14}$$

(see Eq. (2.33) of [1]).

Equation (14) is the culmination, or end point, of the Hamiltonian formulation in [1]. In the discussion after this equation the author emphasizes that "the Hamiltonian constraint is missing in the healthy extended [case]" and the total Hamiltonian is not given as a linear combination of constraints. But upon elimination of second-class constraints, the

Hamiltonian analysis is not complete; to find the number of Degrees of Freedom (DoF) (although it can simplify the search for closure, the elimination of the second-class constraints is not always needed), and to restore gauge invariance using the Dirac conjecture (all first-class constraints generate a gauge symmetry), one must demonstrate the closure of the Dirac procedure for the reduced Hamiltonian.

B. Continuation of Hamiltonian analysis

The reduced total Hamiltonian (14) was obtained under the assumption that the constraints, $p_N = 0$ and $\Theta_2 = 0$, are second class and can be eliminated (i.e. a solution, $N = N(\mathcal{H}_\perp, g_{km})$ can be found). The next step is to check closure of the Dirac procedure in the reduced phase-space, and consider the time development of the secondary constraint (i.e. the PB of $\mathcal{H}_i$ with the reduced total Hamiltonian),

$$\dot{\mathcal{H}}_i = \left\{ \mathcal{H}_i(x), \int d^D y H_T(y) \right\} =$$

$$\left\{ \mathcal{H}_i, \int d^D y N(\mathcal{H}_\perp, g)\left(\mathcal{H}_\perp + \sqrt{g} V(\mathcal{H}_\perp, g)\right) \right\} + \left\{ \mathcal{H}_i, \int d^D y N^k \mathcal{H}_k \right\}. \tag{15}$$

The second PB of (15) is known, and it is proportional to the secondary constraints

$$\left\{ H_i, \int d^D y N^m H_m \right\} = \partial_i N^k H_k + \partial_k \left(N^k H_i \right). \tag{16}$$

But the first PB of (15) is more complicated, and only partial results can be obtained easily, e.g. for the first contribution,

$$\left\{ \mathcal{H}_i(x), \int d^D y N(\mathcal{H}_\perp, g)\mathcal{H}_\perp \right\} = \left\{ \mathcal{H}_i(x), \int d^D y [\mathcal{H}_\perp(y) \right\} N(\mathcal{H}_\perp, g)(y)] +$$

$$\left\{ \mathcal{H}_i(x), \int d^D y [N(\mathcal{H}_\perp, g)(y) \right\} \mathcal{H}_\perp(y)] = \partial_i N(\mathcal{H}_\perp, g)\mathcal{H}_\perp + \ldots, \tag{17}$$

where we used the known result for $\mathcal{H}_\perp$ of (9),

$$\left\{ \mathcal{H}_i(x), \int d^D y f(y)\mathcal{H}_\perp(y) \right\} = \partial_i f(x)\mathcal{H}_\perp(x).$$

To have spatial diffeomorphism gauge invariance, which is claimed to be the gauge symmetry of the healthy extended action, the algebra of constraints must be of a very special form and be closed on the secondary constraints. This imposes severe restrictions on the first PB in (15). If first PB is not zero, then a tertiary constraint arises; if it is proportional

to a momentum constraint (not seems to be the case in (17)), then a generator[4] will be changed, and the gauge transformations will be different. The only result compatible with spatial diffeomorphism is that the first PB in (15) is equal to zero. Is this the case? The answer to this question is important, both for the restoration of gauge symmetry, and for DoF counting. In spite of the complexity of expression (15), the answer is: yes (although it raises a new question, to which we shall return). The first PB of (15) is zero; this is true irrespective of the form of solution (13), and true for all terms of the potential given by (5) (compare the first term in (14) with (13)). Taking $\mathcal{H}_\perp$ from (13) gives

$$N\left(\mathcal{H}_\perp, g\right)\left(\mathcal{H}_\perp + \sqrt{g}V\left(\mathcal{H}_\perp, g\right)\right) = \partial_i\left(N\sqrt{g}\frac{\delta V}{\delta a_i}\right)_{N=N(\mathcal{H}_\perp, g)};$$

therefore, the first PB of (15) is indeed zero, and the total reduced Hamiltonian is actually

$$H_T = \int d^D x\left[N^i\mathcal{H}_i + \dot{N}^i p_i\right]. \tag{18}$$

Note: the author's assertion (in [1]) that the total Hamiltonian of the healthy extended actions is not given as a linear combination of constraints is incorrect. The assumption about the existence of a solution to the second-class constraints leads to a very simple reduced Hamiltonian (18), which has closure on the secondary first-class constraints and a simple constraint algebra (16). One can easily calculate DoF (we provide the result for configurational space) based on the number of constraints and the number of variables in reduced phase-space. After the elimination of lapse, the number of independent fields and the number of first-class constraints, becomes $\frac{(D+1)(D+2)}{2} - 1$ and $2D$, which yields

$$\#DoF = \#fields - \#FCC = \frac{D\left(D-1\right)}{2},$$

according to [14].

Let us find the gauge invariance from the first-class constraints. The number of gauge parameters (and symmetries) is known – it is the same as the number of first-class primary

[4] The generator of spatial diffeomorphism is not just a momentum constraint as it is often presented in the literature. If this were true, then the gauge transformations of lapse and shift functions would be zero, which is not the case, and this is a well known fact from the Lagrangian and Hamiltonian analyses of such actions (e.g. see the generator for GR in ADM variables [12]). In general, according to Dirac "if we are to have any motion at all with a zero Hamiltonian, we must have at least one primary first-class constraint" [13]. This fact is very often neglected in the Hamiltonian analysis of the ADM formulation of GR and other ADM-inspired models.

constraints, p_i. Hamiltonian (18) is just a particular, simplified case of the result known for GR in ADM variables, for which the generator of the gauge transformations was obtained long ago by Castellani [12] (who developed a theorem based upon the Dirac conjecture [13], and developed the procedure of restoration of gauge symmetry, which has been successfully applied to many Hamiltonian formulations of gauge theories). For details of application of this method to GR in ADM variables, we refer the reader to the original paper [12] (some additional details can be found in [15]). The generator of spatial diffeomorphism, G_S, for the total Hamiltonian (18) with algebra of constraints (16) is

$$G_S = \int d^D x \left(\dot{\xi}^i p_i + \xi^i \left(H_i + \partial_i N^j p_j + \partial_j \left(N^j p_i \right) \right) \right). \tag{19}$$

The form of the generator is uniquely defined and reflects the whole algebra of constraints (the generator involves all first-class constrains according to the Dirac conjecture). The transformations are calculated using (keeping convention of [12])

$$\delta_S field = \{field, G_S\}.$$

We obtain the spatial diffeomorphism gauge transformation for two fields,

$$\delta_S N^k = -\xi^j \partial_j N^k + N^j \partial_j \xi^k - \dot{\xi}^k, \tag{20}$$

$$\delta_S g_{km} = -\partial_m \xi^i g_{ik} - \partial_k \xi^i g_{im} - \xi^i \partial_i g_{km}. \tag{21}$$

(field N is not part of the reduced phase-space and generator). Note that N^k does not transform as a vector under spatial diffeomorphism transformations because of the last, extra term in (20). Transformations (20)-(21) are equal to the ones presented in Hořava's original paper [3] (up to an overall sign, and for the standard choice of shift, N^k, in ADM formulation). The choice of shift is very important in Hamiltonian formulation because it affects the constraint algebra [16]. For the non-projectable case (i.e. where lapse is a function, $N(x,t)$, that depends on space and time, not on time alone, i.e. $N(t)$), which is subjected to the healthy extension considered in [1], the lapse should also transform under spatial diffeomorphism (see [3]),

$$\delta_S N = -\xi^i \partial_i N. \tag{22}$$

In [1] a few different results can be found for the transformations of N and a_i. The transformation of lapse under spatial diffeomorphism, according to the second line of Eq.

(2.10), is zero, i.e. $\delta_S N = 0$ (there are no terms with the corresponding gauge parameters). If this were the case, the transformations of a_i would also be zero, $\delta_S a_i = 0$ (because of (7)); but they are given in Eq. (2.11) as $\delta_S a_i = -a_j \partial_i \xi^j$, which is not the transformation of a vector under spatial diffeomorphism that should follow from (7) and (22). The correct transformation of vector a_i appears only later (see Eq. (2.20)),

$$\delta_S a_i = -a_j \partial_i \xi^j - \xi^j \partial_j a_i \ , \tag{23}$$

and in such a case, lapse must transform according to (22).

The Hamiltonian formulation provides an algorithmic way of finding the gauge transformations of all fields (that do not need to be specified *a priori*). To find the gauge transformation of lapse, which is not part of reduced phase-space, one must go back one step to return to the second-class constraint (13), and use the transformation of the metric in (21) (shift is not present in (13)).

C. Restoration of gauge transformations of lapse from the second-class constraint

In gauge theories with second-class constraints, the restoration of gauge symmetries for reduced variables proceeds as follows: the solution to the second-class constraints is an expression for one variable in terms of the other variables that have survived reduction in the Hamiltonian, and for which the gauge transformations are known.

As a simple example, we refer to [2] where one scalar field, A, was eliminated by solving a second-class constraint $A = F(B)$, and the transformations of field B were found from the reduced Hamiltonian: $\delta_S B = -\xi^i \partial_i B$. This information is enough to find the transformation of field A:

$$\delta_S A = \delta_S F(B) = \frac{\delta F(B)}{\delta B} \delta_S B = -\frac{\delta F(B)}{\delta B} \xi^i \partial_i B = -\xi^i \partial_i F(B) = -\xi^i \partial_i A. \tag{24}$$

For the case under consideration, the explicit solution of (13) for N is unknown; but the gauge transformation of this equation, as well as that for lapse, can be found in way similar to that shown in (24) by using equation (13), which one may [1] “presume can be explicitly solved”. To shorten our notation, we shall call the solution to (13) $\tilde{N} \equiv N(\mathcal{H}_T, g_{km})$, and find $\delta_S \tilde{N}$ by using (13) and the known transformations (20)-(21). To restore the gauge transformation, the explicit form of potential $V(g_{ij}, a_i)$ is needed. Let us consider only one

simple term from (5),

$$V\left(g_{ij}, a_i\right) = \sqrt{g} N g^{pq} a_p a_q \tag{25}$$

(for the rest of the contributions the result is the same). With this form of potential, equation (13) becomes

$$-\mathcal{H}_\perp - \sqrt{g} g^{pq} \frac{1}{\tilde{N}^2} \partial_p \tilde{N} \partial_q \tilde{N} + 2\frac{1}{\tilde{N}} \partial_i \left(\sqrt{g} g^{iq} \partial_q \tilde{N}\right) = 0 \tag{26}$$

(where we have substituted the solution, $\tilde{N}$).

Performing variation δ_S of (26) one obtains

$$\begin{gathered}
-\delta_S \mathcal{H}_\perp - \delta_S\left(\sqrt{g} g^{pq}\right) \frac{1}{\tilde{N}^2} \partial_p \tilde{N} \partial_q \tilde{N} + 2\sqrt{g} g^{pq} \frac{1}{\tilde{N}^3} \partial_p \tilde{N} \partial_q \tilde{N} \left(\delta_S \tilde{N}\right) - 2\sqrt{g} g^{pq} \frac{1}{\tilde{N}^2} \partial_p \tilde{N} \partial_q \left(\delta_S \tilde{N}\right) \\
-2\frac{1}{\tilde{N}^2} \partial_i \left(\sqrt{g} g^{iq} \partial_q \tilde{N}\right) \left(\delta_S \tilde{N}\right) + 2\frac{1}{\tilde{N}} \partial_i \delta_S \left(\sqrt{g} g^{iq}\right) \partial_q \tilde{N} + 2\frac{1}{\tilde{N}} \sqrt{g} g^{iq} \partial_i \partial_q \left(\delta_S \tilde{N}\right) = 0.
\end{gathered} \tag{27}$$

To solve this equation for $\delta_S \tilde{N}$, we substitute the known transformation of $\mathcal{H}_\perp$ (as a scalar density):

$$\delta_S \mathcal{H}_\perp = -\partial_i \left(\xi^i \mathcal{H}_\perp\right). \tag{28}$$

And the combination, $\sqrt{g} g^{iq}$, which transforms as

$$\delta_S \left(\sqrt{g} g^{iq}\right) = -\sqrt{g} g^{iq} \partial_p \xi^p + \sqrt{g} \left(g^{qm} \partial_m \xi^i + g^{ip} \partial_p \xi^q\right) - \partial_p \left(\sqrt{g} g^{iq}\right) \xi^p, \tag{29}$$

is presented in a form that is explicitly linear in the gauge parameter and its first-order spatial derivatives. Using (28) and (26), the first term of (27) can be written as

$$-\delta_S \mathcal{H}_\perp = \partial_k \left(\xi^k \left(-\sqrt{g} g^{pq} \frac{1}{\tilde{N}^2} \partial_p \tilde{N} \partial_q \tilde{N} + 2\frac{1}{\tilde{N}} \partial_i \left(\sqrt{g} g^{iq} \partial_q \tilde{N}\right)\right)\right). \tag{30}$$

Equation (27), upon substitution of (29) and (30), allows one to find $\delta_S \tilde{N}$. Note: this is not an equation to find a gauge parameter, it should work for all values of the field-independent vector function, ξ^p. We must find $\delta_S \tilde{N}$, which should be consistent with all orders of gauge parameter (it is linear, but with different order of derivatives). It is clear that the highest order derivative of the gauge parameter (for this part of potential) is two (the second last term in (27)), and the highest order derivative of $\delta_S \tilde{N}$ is also two (the last term in (27)). Therefore, the transformation of $\delta_S \tilde{N}$ should be linear in the gauge parameter (without derivatives), i.e.

$$\delta_S \tilde{N} = X_m \xi^m. \tag{31}$$

To find X_m, we substitute (31) into the last term of (27) and keep only contributions with second-order derivatives of the parameter; and we do the same with the second last term, thus

$$+2\frac{1}{\tilde{N}}\left[\sqrt{g}g^{ip}\partial_i\partial_p\xi^q\right]\partial_q\tilde{N}+2\frac{1}{\tilde{N}}X_m\sqrt{g}g^{iq}\partial_i\partial_q\xi^m=0,$$

which yields

$$X_m=-\partial_m\tilde{N}. \tag{32}$$

Using (31) and steps similar to those in (24), we obtain transformation (22). Of course, the rest of terms in (27), those linear in the gauge parameter and linear in the first-order derivative of the gauge parameter, must be checked for consistency with (31) and (32); but this is not difficult to confirm by a straightforward calculation. We performed this calculation for the simplest part of potential (25), and it is not hard to repeat for the rest of terms in (5).

In the conclusion of [1] one can read the statement: "it would be also extremely useful to find explicit dependence N on $\mathcal{H}_T$ and g". From the transformation properties of a solution, it must be a scalar, and it follows that the explicit dependence of N becomes very restricted – only combinations of $\mathcal{H}_\perp$ and g_{kn} that form a scalar are possible. For example, $\tilde{N}=\frac{\mathcal{H}_\perp}{\sqrt{g}}$, or any function of this combination, $F\left(\frac{\mathcal{H}_\perp}{\sqrt{g}}\right)$, are scalars, but not a solution of (13). In fact, it seems to us that there is no combination, which can be constructed form $\mathcal{H}_\perp$ and g_{kn}, that is simultaneously a scalar and a solution of (13); but we were not able to prove this in general.

III. CONCLUSION. PATHOLOGIES OF HAMILTONIAN FORMULATION OF HEALTHY EXTENSION

The assertion [4] that the canonical structure (Hamiltonian formulation) of the Hořava-type actions, when supplemented by the healthy extension, does not present any problem, and that analysis by Klusoň [1, 2] has confirmed this assertion, are unjustifiably optimistic; the Hamiltonian formulation actually exhibits many pathologies.

The reduced Hamiltonian of the healthy extended Hořava model (18), under the assumption that the second-class constraint can be solved, is too simple, and contrary to the statement of [1], it is linear in constraints. Moreover, after Hamiltonian reduction is performed, one should be able to return to the original Lagrangian or its equivalent form by

performing the inverse Legendre transformation,

$$L = \dot{N}^i p_i + \dot{g}_{ij} p^{ij} - H_T = \dot{g}_{ij} p^{ij} - N^i \mathcal{H}_i. \tag{33}$$

In (33) there is no contribution quadratic in momenta, and the momenta cannot be expressed in terms of velocities, preventing one from returning to the Lagrangian.

The order in which reduction is performed (i.e. to reduce at the Lagrangian level then go to the Hamiltonian, or go from the Lagrangian to the Hamiltonian and then reduce at the Hamiltonian level) is interchangeable [10]. Given the action of the healthy extension (1)

$$S\left(N, N^i, g_{km}\right) = \int dt d^D x \sqrt{g} \left(\frac{1}{N} \tilde{K}_{ij} G^{ijkl} \tilde{K}_{kl} - N E^{ij} G_{ijkl} E^{kl} - N V\left(g_{ij}, a_i\right) \right), \tag{34}$$

(to have an explicit dependence on N, we have introduced $\tilde{K}_{ij} \equiv N K_{ij}$), one may perform a variation of $S\left(N, N^i, g_{km}\right)$ with respect to N to obtain:

$$\frac{\delta S}{\delta N} = -\sqrt{g} \frac{1}{N^2} \tilde{K}_{ij} G^{ijkl} \tilde{K}_{kl} - \sqrt{g} E^{ij} G_{ijkl} E^{kl} - \sqrt{g} V + \frac{1}{N} \partial_i \left(N \sqrt{g} \frac{\delta V}{\delta a_i} \right) = 0, \tag{35}$$

which produces a result similar to (13). Therefore, if the solution of the second-class constraint is assumed to exist, then (35) can also be solved. Performing such a Lagrangian reduction (i.e. substituting solution of (35) $N = N\left(\tilde{K}_{ij} G^{ijkl} \tilde{K}_{kl}, E^{ij} G_{ijkl} E^{kl}, g_{km}\right) = \tilde{N}$ (similar to $N = N\left(\mathcal{H}_\perp, g_{km}\right)$)) into (34), yields

$$S\left(N^i, g_{km}\right) = \int dt d^D x \sqrt{g} \frac{2}{\tilde{N}} \tilde{K}_{ij} G^{ijkl} \tilde{K}_{kl}\ , \tag{36}$$

or in an equivalent form,

$$S\left(N^i, g_{km}\right) = \ S = \int dt d^D x \left(-2\sqrt{g} \tilde{N} E^{ij} G_{ijkl} E^{kl} - 2\sqrt{g} \tilde{N} V\left(g_{ij}, a_i\right) \right)_{a_i = \partial_i \ln \tilde{N}}\ . \tag{37}$$

The Hamiltonian formulation of (36) or (37) should lead to reduced Hamiltonian (18). After performing the Legendre transformation

$$H_T = \dot{N}^i p_i + \dot{g}_{ij} p^{ij} - L\left(N^i, g_{km}\right)$$

and then eliminating the velocities in terms of momenta. To obtain (18), the contributions, which are quadratic in momenta, must disappear from a Lagrangian that is quadratic in velocities; but this is impossible.

In the Hamiltonian formulations of extended Hořava models [1], all problems originate from the assumption that the equation for the differential (second-class) constraint can be

solved, as is the case for second-class algebraic constrains in the first-order formulations of field theories (e.g. Maxwell, Yang-Mills, and the affine-metric formulation of GR); but it is this appearance of such a differential constraint that has been presented as an advantage of the extended formulation. The assumption that a solution of such constraints exists is not specific to extended models, and such differential constraints can be found in all Hamiltonian formulations of Hořava-type models. This important difference and the complications related to it [11] were never mentioned in the literature of the Hamiltonian formulation of Hořava-type actions, with the exception of [7]; although it is good that this difference was recognized, one must perform further analysis before any conclusion can be drawn about the healthiness of the Hamiltonian formulations for models where such constraints arose. Is this a problem that is related to some subtle points of the Hamiltonian formulation of field-theoretical models (e.g. see [17]) or is it an indication of some innate pathologies in the models? This is a question that must be answered.

The healthy extension [4] (its Hamiltonian formulation discussed in papers [1, 2])) was created to cure some problems of Hořava-type models, but the proposed cure leads to side effects which are far more severe than the original illness (if not terminal).

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